On Erdős's elementary method in the asymptotic theory of partitions *

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Abstract

Let $m \geq 1$, and let A be the set of all positive integers that belong to a union of ℓ distinct congruence classes modulo m. Let $p_A(n)$ denote the number of partitions of n into parts belonging to A. It is proved that

$$\log p_A(n) \sim \pi\sqrt{\frac{2\ell n}{3m}}.$$

The proof is based on Erdős's elementary method to obtain the asymptotic formula for the usual partition function $p(n)$.

1 Asymptotic formulas for partition functions

Let $p(n)$ denote the number of partitions of n. Hardy and Ramanujan [7, 6] and, independently, Uspensky [14] discovered the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1}$$

It follows that

$$\log p(n) \sim \pi\sqrt{\frac{2n}{3}}. \tag{2}$$

The proof of (1) uses complex analysis and modular functions. Erdős [2] later discovered an elementary proof of this asymptotic formula; his argument is

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complicated, but uses only estimates for the exponential function and induction from the identity

$$np(n) = \sum_{ka \le n} ap(n - ka).$$

The purpose of this paper is to show that Erdős's method, which is rarely used and almost completely forgotten, is powerful enough to produce asymptotic estimates for many partition functions.

Let A be a nonempty set of positive integers, and let $p_A(n)$ the number of partitions of n into parts belonging to the set A. In this paper we consider sets A that are unions of congruence classes. Let $m \ge 1$, and let $r_1, \ldots, r_\ell$ be integers such that

$$1 \le r_1 < r_2 < \cdots < r_\ell \le m$$

and

$$(r_1, \ldots, r_\ell, m) = 1. \tag{3}$$

Let A be the set of all positive integers a such that $a \equiv r_i \pmod{m}$ for some i. The divisibility condition (3) implies that $p_A(n) \ge 1$ for all sufficiently large integers n. We shall prove that

$$\log p_A(n) \sim \pi \sqrt{\frac{2\ell n}{3m}}.$$

This result is not new; it is contained, for example, in a paper of Meinardus [12] that is heavily analytic. We shall prove this result using only Erdős's elementary method.

Andrews [1, Chapter 6] provides references to asymptotic formulas for various partition functions. Among the few papers that use Erdős's ideas are Freitag [3], Grosswald [5], and Kerawala [10, 11]. Expositions of Erdős's original work can be found in the books of Grosswald [4], Hua [9], and Nathanson [13].

2 Estimates for sums of exponential functions

Lemma 1 *If* $0 \le t \le n$, *then*

$$\sqrt{n} - \frac{t}{2\sqrt{n}} - \frac{t^2}{2n^{3/2}} \le \sqrt{n-t} \le \sqrt{n} - \frac{t}{2\sqrt{n}}.$$

Proof. If $0 \le x \le 1$, then

$$1 - \frac{x}{2} - \frac{x^2}{2} \le (1-x)^{1/2} \le 1 - \frac{x}{2}.$$

The result follows by letting $x = t/n$.

Lemma 2 *If* $x > 0$, *then*

$$\frac{e^{-x}}{\left(1 - e^{-x}\right)^2} < \frac{1}{x^2}.$$

If $0 < x \leq 1$, *then*

$$\frac{e^{-x}}{\left(1-e^{-x}\right)^2} > \frac{1}{x^2} - 2.$$

Proof. The power series expansion for e^x gives

$$\begin{aligned} e^{x/2} - e^{-x/2} &= 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{x}{2}\right)^{2k+1} \\ &= x + x^3 \sum_{k=1}^{\infty} \frac{x^{2k-2}}{(2k+1)!2^{2k}}. \end{aligned}$$

If $x > 0$, then

$$e^{x/2} - e^{-x/2} > x$$

and so

$$\frac{e^{-x}}{\left(1-e^{-x}\right)^2} = \frac{1}{\left(e^{x/2}-e^{-x/2}\right)^2} < \frac{1}{x^2}.$$

If $0 < x \leq 1$, then

$$\begin{aligned} e^{x/2} - e^{-x/2} &< x + x^3 \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \\ &< x + x^3 \\ &< \frac{x}{1-x^2} \end{aligned}$$

and so

$$\frac{e^{-x}}{\left(1-e^{-x}\right)^2} = \frac{1}{\left(e^{x/2}-e^{-x/2}\right)^2} > \left(\frac{1}{x} - x\right)^2 > \frac{1}{x^2} - 2.$$

Lemma 3 *If* $0 \leq q < 1$, *then*

$$\sum_{v=1}^{\infty} v^3 q^v < \frac{6q}{(1-q)^4}.$$

Proof. Differentiating the power series

$$\frac{1}{1-q} = \sum_{v=0}^{\infty} q^v,$$

we obtain

$$\begin{aligned} \frac{1}{(1-q)^2} &= \sum_{v=0}^{\infty} v q^{v-1}, \\ \frac{2}{(1-q)^3} &= \sum_{v=0}^{\infty} v(v-1) q^{v-2}, \end{aligned}$$

$$\begin{aligned}
\frac{6}{(1-q)^4} &= \sum_{v=0}^{\infty} v(v-1)(v-2)q^{v-3} \\
&= \sum_{v=0}^{\infty} (v^3 - 3v(v-1) - v)q^{v-3}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{v=0}^{\infty} v^3 q^v &= \frac{6q^3}{(1-q)^4} + 3q^2 \sum_{v=0}^{\infty} v(v-1)q^{v-2} + q\sum_{v=0}^{\infty} vq^{v-1} \\
&= \frac{6q^3}{(1-q)^4} + \frac{6q^2}{(1-q)^3} + \frac{q}{(1-q)^2} \\
&= \frac{q^3+4q^2+q}{(1-q)^4} \\
&< \frac{6q}{(1-q)^4}.
\end{aligned}$$

Lemma 4 *Let n be a positive integer and let c_1 and ε be positive real numbers. Then*

$$\sum_{k=1}^{\infty} \frac{e^{-\frac{c_1 k}{2\sqrt{n}}}}{1-e^{-\frac{c_1 k}{2\sqrt{n}}}} = O\left(n^{\frac{1}{2}+\varepsilon}\right).$$

Proof. We apply the Lambert series identity (Hardy and Wright [8, Theorem 310])

$$\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{k=1}^{\infty} d(k)q^k,$$

where $0 < q < 1$ and $d(k)$ is the divisor function. Let

$$q = e^{-\frac{c_1 k}{2\sqrt{n}}}.$$

Since

$$d(k) \ll k^{\varepsilon}$$

(Hardy and Wright [8, Theorem 315]), and since $e^{-x} \ll x^{-(1+2\varepsilon)}$ for $x \geq c_1/(2\sqrt{n})$, we have

$$e^{-\frac{c_1 k}{2\sqrt{n}}} \ll \left(\frac{2\sqrt{n}}{c_1 k}\right)^{1+2\varepsilon}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{e^{-\frac{c_1 k}{2\sqrt{n}}}}{1-e^{-\frac{c_1 k}{2\sqrt{n}}}} &= \sum_{k=1}^{\infty} d(k) e^{-\frac{c_1 k}{2\sqrt{n}}} \\
&\ll \sum_{k=1}^{\infty} k^{\varepsilon} \left(\frac{2\sqrt{n}}{c_1 k}\right)^{1+2\varepsilon}
\end{aligned}$$

$$
\begin{aligned}
&= n^{\frac{1}{2}+\varepsilon}\left(\frac{2}{c_1}\right)^{1+2\varepsilon}\sum_{k=1}^{\infty}\frac{1}{k^{1+\varepsilon}} \\
&\ll n^{\frac{1}{2}+\varepsilon}.
\end{aligned}
$$

This completes the proof.

Lemma 5 *Let $m, \ell, r_1, \ldots, r_\ell$ be positive integers such that*

$$1 \leq r_1 < r_2 < \cdots < r_\ell \leq m.$$

Let A be the set of all positive integers a such that

$$a \equiv r_i \pmod{m} \qquad \text{for some } i = 1, \ldots, \ell.$$

Let ϑ be a real number with

$$\vartheta > -1.$$

Let

$$c_0 = \pi\sqrt{\frac{2\ell}{3m}},$$

and

$$c_1 = \left(\sqrt{1+\vartheta}\right)c_0 = \pi\sqrt{\frac{2(1+\vartheta)\ell}{3m}}.$$

Then

$$\sum_{k=1}^{\infty}\sum_{a\in A} a e^{-\frac{c_1 k a}{2\sqrt{n}}} = \frac{n}{1+\vartheta} + O\left(n^{\frac{1}{2}+\varepsilon}\right)$$

for every $\varepsilon > 0$.

Proof. Let

$$q = e^{-\frac{c_1 k}{2\sqrt{n}}}.$$

Then $0 < q < 1$. For $1 \leq r \leq m$, we have

$$
\begin{aligned}
\sum_{v=0}^{\infty}(r+mv)e^{-\frac{c_1 k(r+mv)}{2\sqrt{n}}} &= \sum_{v=0}^{\infty}(r+mv)q^{r+mv} \\
&= mq^r\sum_{v=0}^{\infty}vq^{mv} + rq^r\sum_{v=0}^{\infty}q^{mv} \\
&= \frac{mq^{r+m}}{(1-q^m)^2} + \frac{rq^r}{1-q^m} \\
&= \frac{me^{-\frac{c_1 k(r+m)}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2} + \frac{re^{-\frac{c_1 kr}{2\sqrt{n}}}}{1-e^{-\frac{c_1 km}{2\sqrt{n}}}}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{v=0}^{\infty}(r+mv)e^{-\frac{c_1 k(r+mv)}{2\sqrt{n}}} &= \sum_{k=1}^{\infty}\frac{me^{-\frac{c_1 k(r+m)}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2}+\sum_{k=1}^{\infty}\frac{re^{-\frac{c_1 kr}{2\sqrt{n}}}}{1-e^{-\frac{c_1 km}{2\sqrt{n}}}} \\ &= \sum_{k=1}^{\infty}\frac{me^{-\frac{c_1 k(r+m)}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2}+O\left(n^{\frac{1}{2}+\varepsilon}\right),\end{aligned}$$

since

$$0<\sum_{k=1}^{\infty}\frac{re^{-\frac{c_1 kr}{2\sqrt{n}}}}{1-e^{-\frac{c_1 km}{2\sqrt{n}}}}\le m\sum_{k=1}^{\infty}\frac{e^{-\frac{c_1 k}{2\sqrt{n}}}}{1-e^{-\frac{c_1 k}{2\sqrt{n}}}}\ll n^{\frac{1}{2}+\varepsilon}$$

by Lemma 4.

From the definitions of the constants c_0 and c_1, we obtain the upper bound

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{me^{-\frac{c_1 k(r+m)}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2} &< \sum_{k=1}^{\infty}\frac{me^{-\frac{c_1 km}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2} \\ &< \sum_{k=1}^{\infty}\frac{4n}{c_1^2k^2m} \qquad \text{(by Lemma 2)} \\ &= \frac{4n}{(1+\vartheta)c_0^2m}\sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \frac{4n\pi^2}{6(1+\vartheta)c_0^2m} \\ &= \frac{n}{(1+\vartheta)\ell}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{a\in A}ae^{-\frac{c_1 ka}{2\sqrt{n}}} &= \sum_{i=1}^{\ell}\sum_{k=1}^{\infty}\sum_{v=0}^{\infty}(r_i+mv)e^{-\frac{c_1 k(r_i+mv)}{2\sqrt{n}}} \\ &< \sum_{i=1}^{\ell}\left(\frac{n}{(1+\vartheta)\ell}+O\left(n^{\frac{1}{2}+\varepsilon}\right)\right) \\ &= \frac{n}{1+\vartheta}+O\left(n^{\frac{1}{2}+\varepsilon}\right).\end{aligned}$$

We compute a lower bound as follows:

$$\sum_{k=1}^{\infty}\frac{me^{-\frac{c_1 k(r+m)}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2} > m\sum_{k\le\frac{2\sqrt{n}}{c_1 m}}e^{-\frac{c_1 kr}{2\sqrt{n}}}\left(\frac{e^{-\frac{c_1 km}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 km}{2\sqrt{n}}}\right)^2}\right)$$

$$
\begin{aligned}
&> m \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} e^{-\frac{c_1 k r}{2\sqrt{n}}} \left(\frac{4n}{c_1^2 k^2 m^2} - 2 \right) \qquad \text{(by Lemma 2)} \\
&= \frac{4n}{c_1^2 m} \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{e^{-\frac{c_1 k r}{2\sqrt{n}}}}{k^2} + O\left(\sqrt{n}\right).
\end{aligned}
$$

Since $e^{-x} \ge 1 - x$, we have

$$
\begin{aligned}
\sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{e^{-\frac{c_1 k r}{2\sqrt{n}}}}{k^2} &\ge \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{1}{k^2} \left(1 - \frac{c_1 k r}{2\sqrt{n}} \right) \\
&= \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{1}{k^2} - \frac{c_1 r}{2\sqrt{n}} \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{1}{k} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k > \frac{2\sqrt{n}}{c_1 m}} \frac{1}{k^2} - \frac{c_1 r}{2\sqrt{n}} \sum_{k \le \frac{2\sqrt{n}}{c_1 m}} \frac{1}{k} \\
&= \frac{\pi^2}{6} + O\left(\frac{1}{\sqrt{n}} \right) + O\left(\frac{\log n}{\sqrt{n}} \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{m e^{-\frac{c_1 k (r+m)}{2\sqrt{n}}}}{\left(1 - e^{-\frac{c_1 k m}{2\sqrt{n}}} \right)^2} &> \frac{4n}{c_1^2 m} \left(\frac{\pi^2}{6} + O\left(\frac{1}{\sqrt{n}} \right) + O\left(\frac{\log n}{\sqrt{n}} \right) \right) + O\left(\sqrt{n}\right) \\
&= \frac{n}{(1+\vartheta)\ell} + O\left(\sqrt{n} \log n \right),
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_{k=1}^{\infty} \sum_{a \in A} a e^{-\frac{c_1 k a}{2\sqrt{n}}} &= \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \sum_{v=0}^{\infty} (r_i + mv) e^{-\frac{c_1 k (r_i + mv)}{2\sqrt{n}}} \\
&= \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \frac{m e^{-\frac{c_1 k (r_i + m)}{2\sqrt{n}}}}{\left(1 - e^{-\frac{c_1 k m}{2\sqrt{n}}} \right)^2} + O\left(n^{\frac{1}{2}+\varepsilon} \right) \\
&> \sum_{i=1}^{\ell} \left(\frac{n}{(1+\vartheta)\ell} + O\left(\sqrt{n} \log n \right) \right) + O\left(n^{\frac{1}{2}+\varepsilon} \right) \\
&= \frac{n}{1+\vartheta} + O\left(n^{\frac{1}{2}+\varepsilon} \right).
\end{aligned}
$$

This completes the proof.

The notation $\sum_{ka>n}$ (resp. $\sum_{ka \le n}$) means the sum over all positive integers k and all integers $a \in A$ such that $ka > n$ (resp. $ka \le n$).

Lemma 6 *Let A be a set of positive integers, and let c_1 and N_0 be positive numbers. For every $n \geq 2N_0$,*

$$0 < \sum_{ka > n-N_0} ae^{-\frac{c_1 ka}{2\sqrt{n}}} \ll \frac{1}{\sqrt{n}}.$$

Proof. If $n \geq 2N_0$, then $n - N_0 \geq n/2$. Since

$$e^{-x} \ll \frac{1}{x^6} \qquad \text{for } x > 0,$$

we have

$$\begin{aligned}
\sum_{ka > n-N_0} ae^{-\frac{c_1 ka}{2\sqrt{n}}} &\ll \sum_{ka > n-N_0} a \left(\frac{2\sqrt{n}}{c_1 ka} \right)^6 \\
&\ll n^3 \sum_{ka > n-N_0} \frac{1}{k^6 a^5} \\
&\ll n^3 \sum_{ka > n-N_0} \frac{1}{(ka)^{7/2} k^{5/2} a^{3/2}} \\
&\ll n^3 \sum_{ka > n-N_0} \frac{1}{(n/2)^{7/2} k^{5/2} a^{3/2}} \\
&\ll \frac{1}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \sum_{a \in A} \frac{1}{a^{3/2}} \\
&\ll \frac{1}{\sqrt{n}}.
\end{aligned}$$

Lemma 7 *Let A be a set of positive integers, and let c_1 be a positive number. Then*

$$0 < \sum_{ka \leq n} k^2 a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} \ll n^2.$$

Proof. This is a straightforward computation. We have

$$\begin{aligned}
\sum_{ka \leq n} k^2 a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} &\leq \sum_{k=1}^{n} k^2 \sum_{a \in A} a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} \\
&\leq \sum_{k=1}^{n} k^2 \sum_{v=1}^{\infty} v^3 e^{-\frac{c_1 kv}{2\sqrt{n}}} \\
&\leq 6 \sum_{k=1}^{n} \frac{k^2 e^{-\frac{c_1 k}{2\sqrt{n}}}}{\left(1 - e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^4} \qquad \text{(by Lemma 3)}
\end{aligned}$$

$$
\begin{aligned}
&= 6\sum_{k=1}^{n} \frac{e^{-\frac{c_1 k}{2\sqrt{n}}}}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \frac{k^2}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \\
&< 6\sum_{k=1}^{n} \left(\frac{4n}{c_1^2 k^2}\right) \frac{k^2}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \qquad \text{(by Lemma 2)} \\
&\ll n\sum_{k=1}^{n} \frac{1}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2}.
\end{aligned}
$$

Let

$$x = \frac{c_1 k}{2\sqrt{n}}.$$

If $1 \leq k \leq \sqrt{n}$, then $0 < x \leq c_1/2$ and

$$1 - e^{-x} = \int_0^x e^{-t} dt \geq xe^{-x} \geq xe^{-c_1/2}.$$

It follows that

$$\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2 = \left(1-e^{-x}\right)^2 \geq e^{-c_1} x^2 = \frac{e^{-c_1} c_1^2 k^2}{4n},$$

and so

$$\sum_{1 \leq k \leq \sqrt{n}} \frac{1}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \leq \frac{4e^{c_1} n}{c_1^2} \sum_{1 \leq k \leq \sqrt{n}} \frac{1}{k^2} \ll n.$$

If $k > \sqrt{n}$, then

$$\sum_{\sqrt{n} < k \leq n} \frac{1}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \leq \sum_{\sqrt{n} < k \leq n} \frac{1}{\left(1-e^{-\frac{c_1}{2}}\right)^2} \ll n.$$

Therefore,

$$\sum_{k=1}^{n} \frac{1}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \ll n$$

and

$$\sum_{ka \leq n} k^2 a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} \ll n\sum_{k=1}^{n} \frac{1}{\left(1-e^{-\frac{c_1 k}{2\sqrt{n}}}\right)^2} \ll n^2.$$

This completes the proof.

3 Upper and lower bounds for $\log p_A(n)$

We define $p_A(0) = 1$ and $p_A(-n) = 0$ for all $n \geq 1$. We use k to denote a positive integer, v a nonnegative integer, and a an element of the set A of congruence classes modulo m. The asymptotic formula for $\log p_A(n)$ will be proved by induction from the following classical recursion formula.

Lemma 8 *Let A be a nonempty set of positive integers, and let $p_A(n)$ be the number of partitions of n into parts belonging to A. Then*

$$np_A(n) = \sum_{ka \leq n} ap_A(n - ka).$$

Proof. We enumerate the partitions of n into parts belonging to A as follows:

$$n = a_{i,1} + a_{i,2} + \cdots + a_{i,s_i} \qquad \text{for } i = 1, \ldots, p_A(n).$$

Then

$$np_A(n) = \sum_{i=1}^{p_A(n)} \sum_{j=1}^{s_i} a_{i,j} = \sum_{a \in A} aN(a, n),$$

where $N(a, n)$ is the total number of times that the integer a occurs in the $p_A(n)$ partitions of n. The number of partitions in which the integer a occurs at least k times is $p_A(n - ka)$, and so the number of partitions in which the integer a occurs *exactly* k times is

$$p_A(n - ka) - p_A(n - (k + 1)a).$$

Therefore,

$$N(n, a) = \sum_{k=1}^{\infty} k \left(p_A(n - ka) - p_A(n - (k + 1)a)\right) = \sum_{k=1}^{\infty} p_A(n - ka),$$

and so

$$np_A(n) = \sum_{a \in A} aN(a, n) = \sum_{a \in A} \sum_{k=1}^{\infty} ap_A(n - ka) = \sum_{ka \leq n} ap_A(n - ka),$$

since $p_A(n - ka) = 0$ if $ka > n$. This completes the proof.

Theorem 1 *Let $m, \ell, r_1, \ldots, r_\ell$ be positive integers such that*

$$1 \leq r_1 < r_2 < \cdots < r_\ell \leq m.$$

Let A be the set of all positive integers a such that

$$a \equiv r_i \pmod{m} \qquad \text{for some } i = 1, \ldots, \ell.$$

Then

$$\limsup_{n \to \infty} \frac{\log p_A(n)}{\pi \sqrt{\frac{2\ell n}{3m}}} \leq 1.$$

Proof. Let $0 < \varepsilon < 1/2$,

$$c_0 = \pi\sqrt{\frac{2\ell}{3m}},$$

and

$$c_1 = \left(\sqrt{1+\varepsilon}\right) c_0 = \pi\sqrt{\frac{2(1+\varepsilon)\ell}{3m}}.$$

We shall prove that there exists a constant $K = K(\varepsilon)$ such that

$$p_A(n) \leq K e^{c_1\sqrt{n}} \tag{4}$$

for all nonnegative integers n. This implies that

$$\log p_A(n) \leq \log K + \left(\sqrt{1+\varepsilon}\right) c_0\sqrt{n},$$

and so

$$\frac{\log p_A(n)}{c_0\sqrt{n}} \leq \sqrt{1+\varepsilon} + \frac{\log K}{c_0\sqrt{n}}$$

and

$$\limsup_{n\to\infty} \frac{\log p_A(n)}{c_0\sqrt{n}} \leq 1.$$

Therefore, it suffices to prove (4).

Applying Lemma 5 with $\vartheta = \varepsilon$, we have

$$\sum_{k=1}^{\infty}\sum_{a\in A} a e^{-\frac{c_1 ka}{2\sqrt{n}}} = \frac{n}{1+\varepsilon} + O\left(n^{\frac{1}{2}+\varepsilon}\right).$$

There exists a positive integer $N = N(\varepsilon)$ such that

$$\sum_{k=1}^{\infty}\sum_{a\in A} a e^{-\frac{c_1 ka}{2\sqrt{n}}} < n \tag{5}$$

for all $n \geq N$. We can choose a number $K = K(\varepsilon)$ so that the upper bound (4) holds for all positive integers $n \leq N$. Let $n > N$ and assume that inequality (4) holds for all integers less than n. Then

$$\begin{aligned}
np_A(n) &= \sum_{k=1}^{\infty}\sum_{a\in A} a p_A(n-ka) && \text{(by Lemma 8)}\\
&\leq \sum_{k=1}^{\infty}\sum_{a\in A} aKe^{c_1\sqrt{n-ka}} && \text{(by inequality (4))}\\
&\leq K\sum_{k=1}^{\infty}\sum_{a\in A} a e^{c_1\sqrt{n}-\frac{c_1 ka}{2\sqrt{n}}} && \text{(by Lemma 1)}\\
&= Ke^{c_1\sqrt{n}}\sum_{k=1}^{\infty}\sum_{a\in A} a e^{-\frac{c_1 ka}{2\sqrt{n}}} \\
&< nKe^{c_1\sqrt{n}} && \text{(by inequality (5)).}
\end{aligned}$$

Dividing by n, we obtain (4). This completes the proof.

Note that in Theorem 1 we do not assume that $(r_1, \ldots, r_\ell, m) = 1$.

Theorem 2 *Let $m, \ell, r_1, \ldots, r_\ell$ be positive integers such that*

$$1 \leq r_1 < r_2 < \cdots < r_\ell \leq m,$$

and

$$(r_1, \ldots, r_\ell, m) = 1.$$

Let A be the set of all positive integers a such that

$$a \equiv r_i \pmod{m} \qquad \text{for some } i = 1, \ldots, \ell.$$

Then

$$\liminf_{n \to \infty} \frac{\log p_A(n)}{\pi\sqrt{\frac{2\ell n}{3m}}} \geq 1.$$

Proof. Let $0 < \varepsilon < 1/2$,

$$c_0 = \pi\sqrt{\frac{2\ell}{3m}},$$

and

$$c_1 = \left(\sqrt{1-\varepsilon}\right) c_0 = \pi\sqrt{\frac{2(1-\varepsilon)\ell}{3m}}.$$

The divisibility condition $(r_1, \ldots, r_\ell, m) = 1$ implies that there exists a number N_0 such that $p_A(n) \geq 1$ for all integers $n \geq N_0$. We shall prove that there exists a positive number K such that

$$p_A(n) \geq K e^{c_1\sqrt{n}} \tag{6}$$

for all integers $n \geq N_0$. This implies that

$$\liminf_{n \to \infty} \frac{\log p_A(n)}{\pi\sqrt{\frac{2\ell n}{3m}}} \geq 1.$$

By Lemma 5 (with $\vartheta = -\varepsilon$), Lemma 6, and Lemma 7, there exists a number $N_1 = N_1(\varepsilon) \geq 2N_0$ such that, for all integers $n \geq N_1$,

$$\begin{aligned}
&\sum_{ka \leq n-N_0} a e^{-\frac{c_1 ka}{2\sqrt{n}}} + \frac{c_1}{2n^{3/2}} \sum_{ka \leq n-N_0} k^2 a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} \\
&= \sum_{k=1}^{\infty} \sum_{a \in A} a e^{-\frac{c_1 ka}{2\sqrt{n}}} - \sum_{ka > n-N_0} a e^{-\frac{c_1 ka}{2\sqrt{n}}} + \frac{c_1}{2n^{3/2}} \sum_{ka \leq n-N_0} k^2 a^3 e^{-\frac{c_1 ka}{2\sqrt{n}}} \\
&= \frac{n}{1-\varepsilon} + O\left(n^{\frac{1}{2}+\varepsilon}\right) + O\left(n^{-\frac{1}{2}}\right) + O\left(n^{\frac{1}{2}}\right) \\
&> n.
\end{aligned}$$

We can choose a positive number $K = K(\varepsilon)$ such that $p_A(n)$ satisfies inequality (6) for $N_0 \leq n \leq N_1$.

Let $n > N_1$, and suppose that inequality (6) holds for all integers in the interval $[N_0, n-1]$. We shall prove by induction that this inequality also holds for n. Note that $n - ka \geq N_0$ if $ka \leq n - N_0$. By Lemma 8, we have

$$\begin{aligned}
np_A(n) &= \sum_{ka \leq n} ap_A(n-ka) \\
&\geq \sum_{ka \leq n-N_0} ap_A(n-ka) \\
&\geq K \sum_{ka \leq n-N_0} ae^{c_1\sqrt{n-ka}} \\
&\geq K \sum_{ka \leq n-N_0} ae^{c_1\left(\sqrt{n}-\frac{ka}{2\sqrt{n}}-\frac{k^2a^2}{2n^{3/2}}\right)} && \text{(by Lemma 1)} \\
&= Ke^{c_1\sqrt{n}} \sum_{ka \leq n-N_0} ae^{-\frac{c_1ka}{2\sqrt{n}}} e^{-\frac{c_1k^2a^2}{2n^{3/2}}} \\
&\geq Ke^{c_1\sqrt{n}} \sum_{ka \leq n-N_0} ae^{-\frac{c_1ka}{2\sqrt{n}}} \left(1 - \frac{c_1k^2a^2}{2n^{3/2}}\right) && \text{(since } e^{-x} \geq 1-x\text{)} \\
&= Ke^{c_1\sqrt{n}} \left(\sum_{ka \leq n-N_0} ae^{-\frac{c_1ka}{2\sqrt{n}}} - \frac{c_1}{2n^{3/2}} \sum_{ka \leq n-N_0} k^2a^3e^{-\frac{c_1ka}{2\sqrt{n}}} \right) \\
&> nKe^{c_1\sqrt{n}}.
\end{aligned}$$

Dividing by n, we obtain the lower bound (6). This completes the induction.

Theorem 3 *Let $m, \ell, r_1, \ldots, r_\ell$ be positive integers such that*

$$1 \leq r_1 < r_2 < \cdots < r_\ell \leq m,$$

and

$$(r_1, \ldots, r_\ell, m) = 1.$$

Let A be the set of all positive integers a such that

$$a \equiv r_i \pmod{m} \qquad \text{for some } i = 1, \ldots, \ell.$$

Then

$$\log p_A(n) \sim \pi\sqrt{\frac{2\ell n}{3m}}.$$

Proof. This follows immediately from Theorem 1 and Theorem 2.

The following well–known result is an immediate consequence of Theorem 3.

Theorem 4 *Let $q(n)$ denote the number of partitions of n into distinct parts. Then*

$$\log q(n) \sim \pi\sqrt{\frac{n}{3}}.$$

Proof. Let A be the set of positive odd numbers, that is, the set of positive integers congruent to 1 modulo 2. Euler proved that $q(n) = p_A(n)$, so the result follows immediately from Theorem 3 with $m = 2$ and $\ell = 1$.

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